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A FOLK THEOREM FOR ASYNCHRONOUSLY REPEATED GAMES

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We prove a Folk Theorem for asynchronously repeated games in which the set of players who can move in period t, denoted by I_t, is a random variable whose distribution is a function of the past action choices of the players and the past realizations of I_τ's, $\tau = 1, 2, \ldots, t-1$. We impose a condition, the finite periods of inaction (FPI) condition, which requires that the number of periods in which every player has at least one opportunity to move is bounded. Given the FPI condition together with the standard nonequivalent utilities (NEU) condition, we show that every feasible and strictly individually rational payoff vector can be supported as a subgame perfect equilibrium outcome of an asynchronously repeated game.

KEYWORDS: Asynchronously repeated games, Folk Theorem.

1. INTRODUCTION

THIS PAPER IS CONCERNED WITH determining the set of equilibrium payoffs for asynchronously repeated strategic situations, where players may not be able to change their actions *simultaneously*. To state the result, we prove a Folk Theorem for asynchronously repeated games (with observable actions and discounting) in which the set of players who can move in period t, denoted by I_t, is a random variable whose distribution is a function of the past action choices of the players and the past realizations of I_τ's, $\tau = 1, 2, \ldots, t-1$. We impose a condition, the finite periods of inaction (FPI) condition, which requires that the number of periods in which every player has at least one opportunity to move is bounded. Given the FPI condition together with the nonequivalent utilities (NEU) condition of Abreu, Dutta, and Smith (1994), we show that every feasible and strictly individually rational payoff vector can be supported as a subgame perfect equilibrium outcome of an asynchronously repeated game.

Asynchronous choice structure, which may occur in many real world situations,[2] potentially reduces the set of repeated game equilibria. Indeed, Lagunoff and Matsui (1997) recently showed that, when a pure coordination game is repeated asynchronously[3] and the players are sufficiently patient, the only subgame perfect equilibrium payoff is the one that Pareto dominates all other

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[2]For an example of asynchronous choice structure, see Maskin and Tirole (1988) and the references therein.

[3]By the way, they use the term 'asynchronous' to mean that only one player can move in each period, while we use it to mean that not all the players can move simultaneously in each period. See Section 4 for more discussion on the relationship of our results to Lagunoff and Matsui (1997).

payoffs. This paper takes on this issue, and shows that the Folk Theorem holds for the asynchronous choice environments as well as the standard repeated game environments, given the NEU condition. Thus a failure of the NEU condition is necessary for the Lagunoff-Matsui uniqueness result. In this sense, we provide a common framework in which to understand the standard environments as in Fudenberg and Maskin (1986) or Abreu, Dutta, and Smith (1994) and the asynchronous choice environments as in Lagunoff and Matsui (1997).

In the next section, we introduce asynchronously repeated games, which include both simultaneous-move repeated games and single-move repeated games as special cases. Section 3 contains the Folk Theorems under the assumption that mixed actions are observable.[4] We prove the Folk Theorem for the deterministic case, and then for the stochastic case. Section 4 discusses related literature.

2. THE MODEL

2.1. *The Stage Game*

Let $G = (I, (A_i)_{i=1}^n, (u_i)_{i=1}^n)$ denote a strategic form game where $I = \{1, \ldots, n\}$ is the set of players, A_i is the finite set of pure *actions* for player i, and u_i is the stage game payoff function from $A = \times_{i=1}^n A_i$ to $\mathbb{R}$. The payoff vector $u = (u_1, \ldots, u_n)$ is a function from A to $\mathbb{R}^n$. A *mixed action* α_i for player i is a randomization over A_i. We use $\Delta(A_i)$ to denote the set of mixed actions for i, and $\Delta = \times_{i=1}^n \Delta(A_i)$ to denote the set of mixed action profiles, $\alpha = (\alpha_1, \ldots, \alpha_n)$. The function u can be extended as a function from Δ to $\mathbb{R}^n$ in an obvious way, and is continuous. Following the convention, let V be the convex hull of the set of feasible payoff vectors $\{u(a) : a \in A\}$. Let $\underline{v}_i = \min_{\alpha_{-i}} \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})$ be player i's *minimax value*, and $\underline{\alpha}^i = (\underline{\alpha}_i^i, \underline{\alpha}_{-i}^i)$ be a *minimax profile* against player i. The set $V^* = \{v \in V | v_i \geq \underline{v}_i \text{ for all } i\}$ is called the set of feasible, individually rational payoff vectors, and similarly the set $V^{**} = \{v \in V | v_i > \underline{v}_i \text{ for all } i\}$ is called the set of feasible, strictly individually rational payoff vectors. We normalize utilities to obtain $\underline{v}_i = 0$ for all i.

We assume that the stage game satisfies the nonequivalent utilities (NEU) condition introduced by Abreu, Dutta, and Smith (1994).

CONDITION 1 (Nonequivalent Utilities): *For all i and j in I, there do not exist scalars c, d where $d > 0$ such that $u_i(a) = c + du_j(a)$ for all $a \in A$.*

2.2. *Asynchronously Repeated Games*

We now define asynchronously repeated games. The stage game is repeated in periods $t = 0, 1, \ldots$. In period 0, all players choose their respective actions

[4] It is conceptually straightforward to extend the results to the unobservable mixed action case, although we did not work this out in full detail.

$\alpha = (\alpha_1, \ldots, \alpha_n)$. In period $t \geq 1$, a subset $I_t \subseteq I$ of players (I_t may be $\varnothing$) are able to change their actions $\alpha_{I_t} = \alpha|_{I_t}$. Let I_t be a realization of the 2^I-valued random variable $\tilde{I}_t$ whose distribution is a function of the sequence $I(t-1) = (I_1, \ldots, I_{t-1})$ of past realizations of $\tilde{I}_\tau$'s, $\tau = 1, \ldots, t-1$, and the sequence $\alpha(t-1) = (\alpha^0, \ldots, \alpha^{t-1})$ of past mixed action profiles. We will use $\gamma(t)$ to denote $(I(t), \alpha(t))$. Then $\gamma(t)$ can be regarded as a history at the end of period t. We assume that players learn the sequence $I(t)$ when they are given opportunities to move, and that they know the distributions of the random variables $\{\tilde{I}_t\}_{t=1}^{\infty}$.

This formulation includes quite a large class of repeated strategic situations, both deterministic and stochastic. We name a few:

EXAMPLE 1 (Simultaneous-move Games): $I_t = I$ for all t. This is the environment studied in the standard Folk Theorems.

EXAMPLE 2 (Single-move Games): I_t is a singleton for all t. This is the environment studied in Lagunoff and Matsui (1997), and the related literature cited there.

EXAMPLE 3 (Endogenous Timing with Short-run Commitment): If player i changes his action in period t, then he is committed to it for s periods.

An asynchronously repeated game is a tuple $\Gamma = (G, \{\tilde{I}_t\}_{t=1}^{\infty}, \delta)$, where δ is the common discount factor.

2.3. *Strategies*

A history at the *beginning* of period t, a t-history, is $h(t) = (\alpha^0, I_1, \alpha^1, I_2, \ldots, \alpha^{t-1}, I_t) = (\gamma(t-1), I_t)$. Let $H(t)$ be the set of all t-histories. A strategy for player i is a sequence of functions $\sigma_i = \{\sigma_i^t\}_{t=0}^{\infty}$ such that:

(i) $\sigma_i^0 \in \Delta(A_i)$, and

(ii) for $t \geq 1$,

$$\begin{cases} \sigma_i^t\colon H(t) \to \Delta(A_i) & \text{if } i \in I_t, \\ \sigma_i^t = \alpha_i^{t-1} & \text{otherwise.} \end{cases}$$

Each strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ generates a probability distribution over histories in the obvious way, and consequently generates a distribution over the sequences of the stage-game payoff vectors. Thus if $\{g_i^t\}$ is player i's sequence of stage-game payoffs, his objective in the repeated game is to maximize the expected value of the discounted *average* payoff

$$(1-\delta) \sum_{t=0}^{\infty} \delta^t g_i^t.$$

The equilibrium concept we employ is subgame perfect equilibrium.

3. THE FOLK THEOREMS

3.1. *The Deterministic Case*

We first study the deterministic case. That is, we suppose that $\tilde{I}_t$, for all $t = 1, 2, \ldots,$ assumes a particular $I_t \subseteq I$ with probability 1. We impose the following condition.

CONDITION 2 (Finite Periods of Inaction): *For all $i \in I$, there exists an integer t^i such that: for all t and $\gamma(t)$, $i \in \bigcup_{s=1}^{t^i} I_{t+s}$.*

Let $\bar{t} = \max_{i \in I}\{t^i\}$. Then, Condition 2 implies that there exists a uniform bound $\bar{t}$ such that every player has at least one opportunity to move in any block of $\bar{t}$ periods.

THEOREM 1: *Suppose Condition 1 (NEU) and Condition 2 (FPI) hold. Then, for any payoff vector v in V^{**}, there exists a discount factor $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is a subgame-perfect equilibrium (SPE) outcome of the asynchronously repeated game $\Gamma = (G, \{\tilde{I}_t\}_{t=1}^{\infty}, \delta)$.*

PROOF: Fix $v \in V^{**}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a strategy profile such that $u(\alpha) = v$.[5] By the NEU condition, we can find payoff vectors $x^1, x^2, \ldots, x^n$ with the following properties:

(i) $x^i \gg 0$ for all i (strict individual rationality).
(ii) $x_i^i < v_i$ for all i (target payoff domination).
(iii) $x_i^i < x_i^j$ for all i and j, $i \neq j$ (payoff asymmetry).

Let $\tilde{\alpha}^i$ be such that $u(\tilde{\alpha}^i) = x^i$. Also recall that $\underline{\alpha}^i$ is such that $u_i(\underline{\alpha}^i) = \underline{v}_i = 0$.

Equilibrium Strategy Profile:

We will now present the repeated game strategy profile which will support v as a SPE outcome. We will first list states, and then describe players' actions for each state, and finally describe the transition rule.

1. *States*: In the description of the states, we use, in addition to $i \in I$, variables c and d, which take on nonnegative integers. Variable c keeps track of the severity of crime, while variable d keeps track of the total execution of punishment.

There are (i) state N, which is the normal state; (ii) states $M(i, c, d)$'s, which are the minimaxing states; (iii) states $P(i)$'s, which are the post-minimaxing states; (iv) states $ME(i, c)$'s; and (v) states $MX(i)$'s. As will become clear shortly, states $ME(i, c)$'s and $MX(i)$'s are transition states: State $ME(i, c)$ is a transition state from N to $M(i, c, 1)$, and state $MX(i)$ is a transition state from $M(i, c, d)$ to $P(i)$.

[5] We will assume the availability of public randomization, as is the convention. Consequently we will deal only with mixed strategy profiles explicitly.

2. *Actions*: We now describe the action taken by a typical player $j \in I$ for each state. Note that the description below pertains only to the periods when j has an opportunity to move, that is, to t with $j \in I_t$. (i) State N: Play α_j; (ii) State $M(i,c,d)$: Play $\underline{\alpha}_j^i$; (iii) State $P(i)$: Play $\tilde{\alpha}_j^i$; (iv) State $ME(i,c)$: Play $\underline{\alpha}_j^i$; (v) State $MX(i)$: Play $\tilde{\alpha}_j^i$.

3. *Transition Rule*:[6] The game starts in state N.

I. When in state N:

(A) if the observed action profile in the last period, α^{t-1}, is such that (a) $\alpha_i^{t-1} \neq \alpha_i$, and (b) $\alpha_k^{t-1} = \alpha_k$ for all $k \neq i$: Go to state $ME(i,1)$;

(B) in all other cases: Stay in state N.

II. When in state $ME(i,c)$:

(A) if $\alpha_{-i}^{t-1} = \underline{\alpha}_{-i}^i$: Go to state $M(i,c,1)$;

(B) if α^{t-1} is such that (a) $\alpha_{-i}^{t-1} \neq \underline{\alpha}_{-i}^i$, (b) $\alpha_i^{t-1} \neq \underline{\alpha}_i^i$ when $i \in I_{t-1}$, and (c) $\alpha_k^{t-1} = \underline{\alpha}_k^i$ for all other $k \in I_{t-1} \setminus \{i\}$: Go to state $ME(i,c+1)$;

(C) if α^{t-1} is such that (a) $\alpha_{i'}^{t-1} \neq \underline{\alpha}_{i'}^i$ for $i' \neq i$ and $i' \in I_{t-1}$, and (b) $\alpha_k^{t-1} = \underline{\alpha}_k^i$ for all other $k \in I_{t-1} \setminus \{i'\}$: Go to state $ME(i',1)$;

(D) in all other cases: Stay in state $ME(i,c)$.

III. When in state $M(i,c,d)$:

(A) if α^{t-1} is such that (a) $\alpha_{i'}^{t-1} \neq \underline{\alpha}_{i'}^i$ for $i' \neq i$, and (b) $\alpha_k^{t-1} = \underline{\alpha}_k^i$ for all $k \neq i', i$: Go to state $ME(i',1)$;[7]

(B) in all other cases: If $d < cT$ (T will be determined shortly), then go to state $M(i,c,d+1)$; otherwise go to state $MX(i)$.

IV. When in state $MX(i)$:

(A) if $\alpha^{t-1} = \tilde{\alpha}^i$: Go to state $P(i)$;

(B) if α^{t-1} is such that (a) $\alpha_{i'}^{t-1} \neq \tilde{\alpha}_{i'}^i$ for $i' \in I_{t-1}$, and (b) $\alpha_k^{t-1} = \tilde{\alpha}_k^i$ for all other $k \in I_{t-1} \setminus \{i'\}$: Go to state $ME(i',1)$;

(C) in all other cases: Stay in state $MX(i)$.

V. When in state $P(i)$:

(A) if α^{t-1} is such that (a) $\alpha_{i'}^{t-1} \neq \tilde{\alpha}_{i'}^i$, and (b) $\alpha_k^{t-1} = \tilde{\alpha}_k^i$ for all $k \neq i'$: Go to state $ME(i',1)$;

(B) in all other cases: Stay in state $P(i)$.

Observe that in the last period of state $ME(i,c)$, players play $\underline{\alpha}^i$. Similarly, in the last period of state $MX(i)$, they play $\tilde{\alpha}^i$. This in particular implies that when the game is one of simultaneous moves, state $ME(i,c)$ collapses into state $M(i,c,0)$ and state $MX(i)$ collapses into state $P(i)$.

Choice of Parameter:

Let $b = \max_i \max_a u_i(a)$ and $w = \min_i \min_a u_i(a)$. Since $x^i \gg 0$ for all i, we can choose $\varepsilon > 0$ such that $x_j^i > \varepsilon$ for all i, j. Now choose T to satisfy

$$(*) \qquad \frac{b + 2\bar{t}(b-w)}{T+1} < \frac{\varepsilon}{2}.$$

[6]The transition between the states occurs at the very beginning of each period, before players' actions are taken.

[7]Note that we ignore the deviation by the initial deviator i.

Verification:

We will first show that no one-shot deviation from the specified strategy by any player in any state is profitable for large enough δ. First, let us define

$$V^d(k,c+1) = (1-\delta)b + \delta\Big[(1-\delta^{\bar{t}})b + \delta^{\bar{t}+(c+1)T}(1-\delta^{\bar{t}})b + \delta^{2\bar{t}+(c+1)T}x_k^k\Big].$$

$V^d(k,c+1)$ is the maximal lifetime (discounted average) payoff the deviator, player k, gets when the severity of crime is given by $c+1$.

I. State $ME(i,c)$:

(A) Player i: i gets at least $V^1 = (1-\delta^{\bar{t}})w + \delta^{\bar{t}+cT}(1-\delta^{\bar{t}})w + \delta^{2\bar{t}+cT}x_i^i$ if he conforms. Thus, it is better to conform if $V^1 > V^d(i,c+1)$. That is, player i will not deviate if

$$x_i^i > \frac{(1-\delta)b + (1-\delta^{\bar{t}})(\delta b - w) + (1-\delta^{\bar{t}})\delta^{\bar{t}+cT}(\delta^{T+1}b - w)}{\delta^{2\bar{t}+cT}(1-\delta^{T+1})}.$$

The right-hand side goes to $(b+2\bar{t}(b-w))/(T+1)$ as $\delta \to 1$ by l'Hôpital's Rule. By (*), we thus have the right-hand side $< \varepsilon$ for sufficiently large δ. Since $x_i^i > \varepsilon$ by construction, player i will not deviate.

(B) Player j ($j \neq i$): j gets at least $V^2 = (1-\delta^{2\bar{t}+cT})w + \delta^{2\bar{t}+cT}x_j^i$ if he conforms. Thus, it is better to conform if $V^2 > V^d(j,1)$, i.e.,

$$\left(\delta^{2\bar{t}+cT}x_j^i - \delta^{2\bar{t}+T+1}x_j^j\right) > (1-\delta)b + \delta[(1-\delta^{\bar{t}})b + \delta^{\bar{t}+T}(1-\delta^{\bar{t}})b] - (1-\delta^{2\bar{t}+cT})w.$$

Since the right-hand side $\to 0$ and the left-hand side $\to x_j^i - x_j^j > 0$ (payoff asymmetry) as $\delta \to 1$, player j will not deviate for sufficiently large δ.

II. State $M(i,c,d)$:

(A) Player i: Since i is minimaxed and his action in this state is ignored, he has no incentive not to play $\underline{\alpha}_i^i$.

(B) Player j ($j \neq i$): j gets at least $V^3 = (1-\delta^{\bar{t}+cT})w + \delta^{\bar{t}+cT}x_j^i$ if he conforms. Thus, it is better to conform if $V^3 > V^d(j,1)$, i.e.,

$$\left(\delta^{\bar{t}+cT}x_j^i - \delta^{2\bar{t}+T+1}x_j^j\right) > (1-\delta)b + \delta[(1-\delta^{\bar{t}})b + \delta^{\bar{t}+T}(1-\delta^{\bar{t}})b] - (1-\delta^{\bar{t}+cT})w.$$

Since the right-hand side $\to 0$ and the left-hand side $\to x_j^i - x_j^j > 0$ (payoff asymmetry) as $\delta \to 1$, player j will not deviate for sufficiently large δ.

III. State $MX(i)$:

(A) Player i: It is better to conform if $(1-\delta^{\bar{t}})w + \delta^{\bar{t}}x_i^i > V^d(i,1)$, i.e.,

$$x_i^i > \frac{(1-\delta)b + \delta[(1-\delta^{\bar{t}})b + \delta^{\bar{t}+T}(1-\delta^{\bar{t}})b] - (1-\delta^{\bar{t}})w}{\delta^{\bar{t}}(1-\delta^{\bar{t}+T+1})}.$$

Since $\lim_{\delta\to 1}$ the right-hand side $= (b+\bar{t}(2b-w))/(T+\bar{t}+1) < \varepsilon/2$ by (*) (note that $w \leq 0$), player i will not deviate for sufficiently large δ.

(B) Player j $(j \neq i)$: It is better to conform if $(1-\delta^{\bar{t}})w + \delta^{\bar{t}} x_j^i > V^d(j,1)$, i.e.,

$$\left(\delta^{\bar{t}} x_j^i - \delta^{2\bar{t}+T+1} x_j^j\right) > (1-\delta)b + \delta[(1-\delta^{\bar{t}})b + \delta^{\bar{t}+T}(1-\delta^{\bar{t}})b] - (1-\delta^{\bar{t}})w.$$

Since the right-hand side $\to 0$ and the left-hand side $\to x_j^i - x_j^j > 0$ (payoff asymmetry) as $\delta \to 1$, player j will not deviate for sufficiently large δ.

IV. State $P(i)$:

(A) Player i: It is better to conform if $x_i^i > V^d(i,1)$. Since $x_i^i > (1-\delta^{\bar{t}})w + \delta^{\bar{t}} x_i^i$, the conclusion that i will not deviate for sufficiently large δ follows directly from case III(A).

(B) Player j $(j \neq i)$: It is better to conform if $x_j^i > V^d(j,1)$. Since $x_j^i > x_j^j > V^d(j,1)$ by payoff asymmetry and case IV(A), the conclusion that j will not deviate for sufficiently large δ holds.

V. State N: It is better to conform if $v_i > V^d(i,1)$. Since $v_i > x_i^i > V^d(i,1)$ by target domination and case IV(A), the conclusion that i will not deviate for sufficiently large δ holds.

We have shown that no player has an incentive to make a one-shot deviation from any state for sufficiently large δ. Now, by Condition 2, which ensures that the play will move from state $ME(i,c)$ to state $M(i,c,1)$ after $\bar{t}$ periods at the latest, we know that there is no profitable sequence of deviations since we can choose δ large enough to satisfy the inequality in case I(A) for $c = 1, \ldots, \bar{t}$. In other words, it is impossible that a player deviates again and again so that the play reaches a level of c in $ME(i,c)$ for which the given δ may not be sufficient to prevent a one-shot deviation. This concludes the proof. *Q.E.D.*

We use variable c, which keeps track of the severity of crime, to control the deviator's incentive to deviate again in the transition phase $ME(i,c)$: Due to inertia, the deviator, say player i, may have an opportunity to move again before other players convert to the minimax profile $\underline{\alpha}_{-i}^i$. To make player i conform to the prescribed action $\underline{\alpha}_i^i$, and thus to ensure that the play moves to the minimaxing phase $M(i,c,1)$, we need to devise a penal code in which the deviator who deviates again in the transition phase $ME(i,c)$ is punished more severely (by lengthening the minimaxing phase). Therefore, Abreu's (1988) *simple penal codes* cannot be used here given the way that the rest of the equilibrium strategies have been constructed. Note, however, that Abreu-style simple penal codes are applied for other players and other phases. That is, the same (possibly player-specific) punishment scheme is applied for other players and other phases (other than the deviator in the transition phase $ME(i,c)$), regardless of the period and the situation in which the deviation occurs. It remains an open question whether Theorem 1 can be proved using stationary punishments.

3.2. *The Stochastic Case*

In this subsection, we restore our general framework that the set of players who can move in period t, i.e., $\tilde{I}_t$, is a random variable. We first observe that

Theorem 1 can be applied to this case if we replace Condition 2 by the following condition: For all $i \in I$, there exists an integer t^i such that $\Pr(i \in \bigcup_{s=1}^{t^i} \tilde{I}_{t+s} | t, \gamma(t)) = 1$ for all t and $\gamma(t)$. Now, to obtain the Folk Theorem for richer classes of asynchronously repeated games, let us define $\tilde{\tau}(t, \gamma(t)) = \min_{t'}\{I \subseteq \bigcup_{s=1}^{t'} \tilde{I}_{t+s}\}$. $\tilde{\tau}(t, \gamma(t))$ is the minimum number of periods in which every player has at least one opportunity to move, given the history $\gamma(t)$ at the end of period t. We will assume that $\tilde{\tau}(t, \gamma(t))$'s are independent and have a common finite mean.

CONDITION 3 (Finite Periods of Inaction in Expectation): (i) $\tilde{\tau}(t, \gamma(t))$'s are independent, and (ii) $E[\tilde{\tau}(t, \gamma(t))] = \bar{t}$, for all t and $\gamma(t)$.

Condition 3 imposes a stationary structure on the repeated game. Note also that, with a positive probability, some players may have no opportunity to move in any finite interval of periods. As a simple example which satisfies the condition, consider a two-player game in which only one of the players is randomly given an opportunity to move in each period. If each player is given an opportunity with equal probability, then $E[\tilde{\tau}] = \sum_{n=1}^{\infty}(n+1)/2^n = 3 < \infty$. We have the following theorem.

THEOREM 2: *Suppose Condition* 1 (*NEU*) *and Condition* 3 (*FPIE*) *hold. Then, for any payoff vector* v *in* V^{**}, *there exists a discount factor* $\underline{\delta} < 1$ *such that, for all* $\delta \in (\underline{\delta}, 1)$, v *is a subgame-perfect equilibrium* (*SPE*) *outcome of the asynchronously repeated game* $\Gamma = (G, \{\tilde{I}_t\}_{t=1}^{\infty}, \delta)$.

PROOF: The proof is basically the same as that of Theorem 1: The states, the actions, the transition rule, and the choice of parameters are the same. The differences arise in the verification part. Let us define

$$W^d(k, c+1) = E\Big[(1-\delta)b + \delta\big[(1-\delta^{t_1})b + \delta^{t_1+(c+1)T}(1-\delta^{t_2})b + \delta^{t_1+t_2+(c+1)T}x_k^k\big]\Big],$$

where the expectation is taken over t_1, which is the number of periods the play stays in $ME(k, c+1)$, and t_2, which is the number of periods the play stays in $MX(k)$. $W^d(k, c+1)$ is the maximal expected lifetime (discounted average) payoff the deviator, player k, gets when the severity of crime is given by $c+1$. Below we will demonstrate how the proof of Theorem 1 could be adapted by verifying player i's incentive in state $ME(i, c)$. In $ME(i, c)$, player i gets at least $W^1 = E[(1-\delta^{t_3})w + \delta^{t_3+cT}(1-\delta^{t_4})w + \delta^{t_3+t_4+cT}x_i^i]$ in expectation if he conforms. In the expression, the expectation is taken over t_3, which is the number of periods the play stays in $ME(i, c)$, and t_4, which is the number of periods the play stays in $MX(i)$. Thus, it is better to conform if $W^1 > W^d(i, c+1)$. That is, player i will not deviate if

$$x_i^i > \frac{\begin{array}{r}(1-\delta)b + \delta E[(1-\delta^{t_1})b + \delta^{t_1+(c+1)T}(1-\delta^{t_2})b] \\ -E[(1-\delta^{t_3})w + \delta^{t_3+cT}(1-\delta^{t_4})w]\end{array}}{\delta^{cT}E[\delta^{t_3+t_4} - \delta^{t_1+t_2+T+1}]}.$$

The right-hand side goes to $(b + 2\bar{t}(b-w))/(T+1)$ as $\delta \to 1$ by the Dominated Convergence Theorem, l'Hôpital's Rule, and Condition 3. By (*) in the proof of Theorem 1, we thus have the right-hand side $< \varepsilon$ for sufficiently large δ. Since $x_i^i > \varepsilon$ by construction, player i will not deviate. The rest of the proof proceeds in a similar way. *Q.E.D.*

4. DISCUSSION

As explained in the introduction, our result is closely related to the results of Lagunoff and Matsui (1997). They proved that when a pure coordination game is repeated so that no two players can move simultaneously at any point in time, the only subgame perfect equilibrium payoff is the one that Pareto dominates all other payoffs. They prove this uniqueness result for "general asynchronously repeated games," which is defined by using *semi-Markov processes*.[8] Our stochastic environments are restrictive in two ways compared to their environments. First, we only study discrete-period environments, not the continuous-time environments. Second, we have the FPIE condition (Condition 3), which imposes a certain finite mean restriction on the renewal process together with a certain symmetry condition, which roughly requires that the expectation of the event that all the states are visited is the same regardless of the state the process starts, on the Markov chain of the semi-Markov process. On the other hand, while their uniqueness result is only for single-move games, our Folk Theorem results hold for general nonsimultaneous-move games, by which we mean repeated games in which not all the players move simultaneously in each period, as well as for simultaneous-move games.

Another result worth noting in this context is the Folk Theorem of Dutta (1995). Dutta studied stochastic games in which a state variable represents the environment of the game and its evolution is determined by the initial conditions, players' actions, and the transition law. With appropriate assumptions on the set $F(s)$ of feasible long-run average payoffs and the long-run average minimax value $m_i(s)$, together with the standard full dimensionality condition, he proved a Folk Theorem. Since asynchronously repeated games may be regarded as stochastic games by treating the set 2^I as the set S of states, an explanation of the differences is in order. First, the framework in this paper does not strictly fit into the stochastic games framework because, while Dutta's law of motion q is stationary across periods, the random variables $\{\tilde{I}_t\}_{t=1}^{\infty}$ in this

[8]A semi-Markov process is a stochastic process that makes transition from state to state in accordance with a Markov chain, but in which the amount of time spent in each state before a transition occurs is random and follows a renewal process.

paper may not be stationary. Next, the set $F(s)$ need not coincide with the convex hull V of the set of feasible payoff vectors of the stage game, nor need $m_i(s)$ coincide with the minimax value $\underline{v}_i$. Indeed, the set $F(s)$ and the value $m_i(s)$ in stochastic games framework are affected by the law of motion itself, while the set V and the value $\underline{v}_i$ in this paper are not.[9] While Dutta imposes conditions directly on $F(s)$ and $m_i(s)$, we impose conditions on the primitives of the model, which gives us a better understanding of the nature of dynamic interaction.

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[9]For example, if one player is committed to a particular action forever, then $F(s)$ loses one dimension and $m_i(s)$ is affected correspondingly. Note, however, that Dutta's results are consistent with ours when reduced to a common framework.