A Folk Theorem under Anonymity *

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Abstract

We extend the classic Folk Theorem of Fudenberg and Maskin(1986) by introducing anonymity in the environment. In doing so, we show that anonymity alone cannot affect the scope for cooperation. We construct a punishment scheme which correctly pinpoints the deviator and punishes him and only him for finite periods. This scheme is not particularly more demanding than the usual punishment schemes for the environment without anonymity. In this sense, this paper conceptually separates anonymity from other aspects of repeated situations, like the decentralized information transmission process or noisy observation, and show that anonymity cannot be a culprit for undesirable outcomes.

Keywords: Folk Theorem, Anonymity, Repeated Games, Cooperation

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1. INTRODUCTION

Anonymous interactions are one of the most distinct aspects of modern societies. Cities, for example, consist of a large number of people who do not and need not know the identities of their neighbors, but still constantly interact with each other. In addition, various legal restrictions also contribute to the degree of anonymity.

Anonymity may produce undesirable social outcomes, as suggested by casual observations and documented evidences.[1)] Under anonymity, a person may act more irresponsibly and selfishly than when his identity is revealed, since (i) he knows that he will not be identified (that is, there does not exist any fear of direct punishment), and (ii) the consequences of his action will be assumed by the society as a whole, not by himself (that is, the responsibility is diffused to all members of society). Therefore, cooperation among the members may be extremely hard, if not impossible, to achieve.

Contrary to this common perception, we prove in this paper that anonymity alone does not affect the scope for cooperation. We construct a punishment scheme which correctly pinpoints the deviator and punishes him and only him for finite periods so that, on the one hand, any possible gains from deviation are wiped out and, on the other hand, the society will eventually return to cooperation. This scheme is *not* severely more demanding than the usual punishment schemes for repeated games with discounting where anonymity is not assumed, in the sense that we do not require more patient players (i.e., higher discount factor δ).

This result can be contrasted with the Folk Theorems of random matching models, which are also concerned with the question of supporting cooperation under anonymous social settings.[2)] These models assume, besides anonymity, the decentralized information

1) See, for example, Milgram (1970) or Neal (1993) and references therein.

2) Rosenthal (1979), Rosenthal and Landau (1979), Okuno-Fujiwara and Postlewaite (1990), Kandori (1992), Harrington (1995), and Ellison (1994).

transmission process. In these models, players are randomly matched in pairs and each player *cannot* observe the action choices of players outside of his own matching nor even the identities of his past and present partners. Because of this special structure of the information transmission process, a different kind of punishment scheme is used to support cooperation. In the *contagious equilibrium* of these models, whenever a person observes a deviation then he himself deviates from then on. Hence a deviation spreads through the society like the flu, and cooperation (or the norm, as is usually called) will eventually collapse. Therefore, each player has a strong incentive not to initiate a deviation, and this is the logic behind the Folk Theorems of these models.

Observe that, in this equilibrium, the punishment is not directed only to the deviator; the society as a whole assumes the consequences of the deviation since everybody in the society will suffer from the collapse of the norm. Also, as perceived by the authors of these models, this strategy has a drawback in that it is too fragile. A small amount of noise or mistakes would lead to the complete collapse of the norm. In addition, due to the complicated strategic situations arising from the decentralized information transmission process, relatively modest progress has been made in this area. A Folk Theorem holds if the stage game has a dominant action. (The Prisoners' Dilemma is one such game.) But at present we don't know whether a Folk Theorem will hold or not for more general games.

In contrast, we prove a Folk Theorem in this paper which holds for more general games. Specifically, every feasible, strictly individually rational payoff vector can be supported as a sequential equilibrium outcome when the stage game satisfies a certain symmetry assumption.[3)] Also, as noted earlier, only the deviator will be punished and the society will resume cooperation after the punishment. This result is due to the theoretically more tractable structure that information is transmitted in some centralized way. That is, even though players can't observe the identities of other players, they can still learn what is happening in the society without much delay. We provide explicit repeated game strategies

[3)] See Assumption 1 in the next section and the discussion following it.

for both the observable and the unobservable mixed action cases, which are more or less the standard strategies found in the usual Folk Theorems. In fact, we follow the proofs of standard Folk Theorems (as in Fudenberg and Maskin (1986) or Abreu, Dutta, and Smith (1994)) as close as possible to demonstrate that essentially the same strategies can be used even under anonymity. In this respect, this paper is a re-interpretation of the Folk Theorems.

In the next section, we present the model. We first prove a Folk Theorem with the assumption that mixed actions are observable, which is the content of section 3. In section 4, we relax the observability assumption and suppose that only the actual realizations of mixed actions are observable, and prove a Folk Theorem. Discussion of the related literature is given in section 5. Note especially the discussion of the relationship between the present paper and Fudenberg, Levine, and Maskin (1994).

2. THE MODEL

There are $n(n \geq 3)$ players. In each period t $(t = 0, 1, \ldots)$, players play the *stage game* $G = \big(I, (A_i)_{i=1}^n, (u_i)_{i=1}^n\big)$ (where I is the set of players, A_i is the finite set of pure *actions* for player i, and u_i is the stage game payoff function from $A = \times_{i=1}^n A_i$ to $\mathbb{R}$). Let $u = \times_{i=1}^n u_i$. A *mixed action* α_i for player i is a randomization over A_i. Let $\Delta(A_i)$ denote the set of mixed actions for i. And let $\Delta = \times_{i=1}^n \Delta(A_i)$ denote the set of mixed action profiles, with a typical element $\alpha = (\alpha_1, \ldots, \alpha_n)$. The function u can be extended as a function from Δ to $\mathbb{R}^n$ in the obvious way, and is continuous. Two points are worth mentioning: First, the set $\Delta(A_i)$ naturally includes pure actions. Second, we can alternatively think of $\Delta(A_i)$, which is compact and convex, as the set of player i's pure actions.

Following the convention, let V be the convex hull of the set of feasible payoff vectors $\{u(a) : a \in A\}$. Let $\underline{v}_i = \min_{\alpha_{-i}} \max_{\alpha_i} u_i(\alpha_i, \alpha_{-i})$ be player i's *minimax value*, and $\underline{\alpha}^i = (\underline{\alpha}_i^i, \underline{\alpha}_{-i}^i)$ be a *minimax profile* against player i. The set $V^* = \{v \in V | v_i \geq \underline{v}_i$ for all $i\}$ is called the set of feasible, individually rational payoff vectors, and similarly the set

$V^{**} = \{v \in V | v_i > \underline{v}_i \text{ for all } i\}$ is called the set of feasible, strictly individually rational payoff vectors. We normalize $\underline{v}_i = 0$ for all i. We will assume the availability of public randomization, as is the convention. Consequently we will deal only with mixed action profiles explicitly. This practice is not restrictive. Fudenberg and Maskin (1991) prove that players can replace the public randomization with a deterministic sequence of actions.

We assume that the game is symmetric and anonymous in the sense defined below. For that, let us first introduce some notation: Let X be a finite set. By a permutation π of X, we mean a bijection $\pi : X \to X$. We extend this notation from sets to ordered k-tuples in the following natural way. If $y = (y_1, \ldots, y_k)$ and if π is a permutation of $\{1, \ldots, k\}$, then we say that $(y_{\pi(1)}, \ldots, y_{\pi(k)})$ is a permutation of y and denote it by $\pi(y)$. In particular,

Definition 1: Let $J \subseteq I$ be a subset of players with m members, and π be a permutation of J.

(a) Given a mixed action profile $\alpha_J = (\alpha_{j(1)}, \ldots, \alpha_{j(m)})$ for J, a *permutation* $\pi(\alpha_J)$ of α_J is $(\alpha_{\pi(j(1))}, \ldots, \alpha_{\pi(j(m))})$.

(b) Given a payoff vector $v_J = (v_{j(1)}, \ldots, v_{j(m)})$ for J, a *permutation* $\pi(v_J)$ of v_J is $(v_{\pi(j(1))}, \ldots, v_{\pi(j(m))})$.

Assumption 1: (Symmetry)

(1) For all i, j in I, we have $A_i = A_j$.

(2) For all $a \in A$, and for all permutation $\pi(a)$ of a,

$$u_i(\pi(a)) = u_{\pi(i)}(a).$$ [4]

Assumption 1(2) says that if player i acts like player $\pi(i)$ (i.e., i's action is $\pi(a_i)$) for all i, then i's payoff from that action profile is equal to the $\pi(i)$'s payoff when players play the action profile a.

[4] This can be weakened so that we only require $u_i(\pi(a)) = c_{i,\pi(i)} u_{\pi(i)}(a) + d_{i,\pi(i)}$ where $c_{i,\pi(i)}$ and $d_{i,\pi(i)}$ are constants and $c_{i,\pi(i)} > 0$. That is, an affine transformation of utilities can be taken care of.

Assumption 1 can be relaxed to include more general games in such a way that the players are divided into k subgroups (that is, there are k different *types* of players) and the symmetry assumption is imposed on each subgroup, as long as the actions available to one group are distinct from the actions of other groups. One example is the case when there are two groups, buyers and sellers, and buyers (sellers) are identical among themselves. Since this extension is straightforward conceptually, but adds unnecessary complication in stating and deriving the results, we will work only with the simplest case.

For a given action profile $\alpha \in \Delta$, let

$$\begin{aligned} P(\alpha) =& \{\alpha' \in \Delta : \alpha' \text{ is a permutation of } \alpha\} \\ =& \{\pi(\alpha) : \pi \text{ is a permutation of } I\}. \end{aligned}$$

Similarly, for $\alpha_{-i} \in \Delta_{-i}$, let

$$\begin{aligned} P(\alpha_{-i}) =& \{\alpha'_{-i} \in \Delta_{-i} : \alpha'_{-i} \text{ is a permutation of } \alpha_{-i}\} \\ =& \{\pi(\alpha_{-i}) : \pi \text{ is a permutation of } I - \{i\}\}. \end{aligned}$$

Then we have

Lemma 1: $\forall i, \forall \alpha = (\alpha_i, \alpha_{-i})$,

$$u_i(\alpha_i, \alpha_{-i}) = u_i(\alpha_i, \alpha'_{-i}) \text{ for all } \alpha'_{-i} \in P(\alpha_{-i}).$$

Proof: For any permutation $\alpha'_{-i} \in P(\alpha_{-i})$, let $\alpha' = (\alpha_i, \alpha'_{-i})$. Then α' is a permutation of α with $\pi(i) = i$. Thus, by Assumption 1, $u_i(\alpha') = u_i(\pi(\alpha)) = u_{\pi(i)}(\alpha) = u_i(\alpha)$. *Q.E.D.*

Lemma 2: *V is symmetric, i.e.,*

$$v = (v_1, \ldots, v_n) \in V \Rightarrow \pi(v) \in V$$

where π is any permutation of I.

Proof: Let α be an action profile with $u(\alpha) = v$. Then $\pi(\alpha)$ will give $\pi(v)$. To be more explicit, we want to show $(v_{\pi(1)}, \ldots, v_{\pi(n)}) \in V$ whenever $(v_1, \ldots, v_n) \in V$. Think of an action profile in which player i plays $\alpha_{\pi(i)}$ for all $i = 1, \ldots, n$. Then we have

$u_i(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) = u_{\pi(i)}(\alpha_1, \ldots, \alpha_n) = v_{\pi(i)}$ by Assumption 1. So, $(v_{\pi(1)}, \ldots, v_{\pi(n)}) \in V$. *Q.E.D.*

Assumption 2: (Anonymity) Players observe the action profile at the end of each period, but do not observe the identities of other players.

Thus, Assumption 2 means that player i cannot distinguish among the elements of $P(\alpha_{-i})$. Note that we assume players observe the mixed action profile, not the realization of it. As we mentioned in the introduction, this assumption will be maintained up to the end of section 3. We can conceive of a situation where, although players cannot tell the identity of a player who played a particular action by observing the action profile, each player is still able to do that by looking at his or her own payoff. This is ruled out by Lemma 1. Therefore, our formulation is the extreme and simple case to analyze the effect of anonymity. (The intermediate cases could as well be analyzed without much difficulty. The conclusion would not be affected.)

Assumption 2 in the present form is not necessary. What is essentially needed is that a deviation will be *accurately* detected by all players (although they don't know exactly who the actual deviator is). So, there are many alternative ways to incorporate this requirement. For example, we may assume that players observe some aggregate variable which will be determined by the action profile.

Now let us turn in detail to the *repeated game.* In each period t, the stage game is played, resulting in a *public outcome*[5] $y^t = [\alpha_1^t, \ldots, \alpha_n^t]$, where the notation $[\ldots]$ signifies the fact that the order is not relevant. The *public history* at the end of period t is $h^t = (y^0, \ldots, y^t)$. Player i's *private history* at the end of period t is $h_i^t = (\alpha_i^0, \ldots, \alpha_i^t)$. A strategy σ_i for player i is a sequence of functions $\{\sigma_i^t\}_{t=0}^{\infty}$, where $\sigma_i^0 \in \Delta(A_i)$ and σ_i^t for $t \geq 1$ maps each pair (h^{t-1}, h_i^{t-1}) to an element of $\Delta(A_i)$.

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5) "Public outcome" here may also be called an "action profile" if we agree that only the collection of actions itself but not the order of them is relevant. In particular, it should not be thought of as a payoff vector.

Each strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ generates a probability distribution over histories in the obvious way, and consequently generates a distribution over the sequences of the stage-game payoff vectors. Players discount future payoffs with a common discount factor δ. Thus if $\{g_i^t\}$ is player i's sequence of stage-game payoffs, his objective in the repeated game is to maximize the expected value of the *average* payoff

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t g_i^t.$$

Due to the informational restriction we gave (i.e. anonymity), each information set is not a singleton. That implies that the repeated game has no proper subgame, and so subgame-perfection is too weak a solution concept to employ. Instead, we will use the *sequential equilibrium* concept (Kreps and Wilson (1982)) in this paper[6)]: A sequential equilibrium is a strategy profile σ together with a belief system μ such that each player's strategy is sequentially rational and the belief system is consistent. Now we impose the following condition on players' payoffs.

Assumption 3: (Non-equivalent payoffs) Not all players have equivalent payoffs, i.e., there exist $k, l \in I$ $(k \neq l)$ such that

$$u_k(a) \neq u_l(a) \text{ for some } a \in A.$$

It is well known that, even without anonymity, we have to impose certain condition on payoffs to get a Folk Theorem for the case of three or more players. Fudenberg and Maskin (1986) introduced the *full dimensionality* condition: The set V^{**} of feasible, strictly individually rational payoff vectors must have dimension n (the number of players), or equivalently, a nonempty interior. Abreu, Dutta, and Smith (1994) weakened this condition to the *non-equivalent utilities* (NEU) condition. A stage game G satisfies the NEU condition if for all i and j in I, there do not exist scalars c, d where $d > 0$ such that $u_i(a) = c + du_j(a)$ for all $a \in A$.

6) If we interpret the compact set $\Delta(A_i)$ as the set of pure strategies, then there arises a problem in defining the interior of the consistent beliefs. In that case, we can instead use a weaker concept, e.g. perfect Bayesian equilibrium concept (Fudenberg and Tirole (1991)).

Assumption 3 is apparently weaker than the NEU condition. On the other hand, we show in Proposition 1 below that Assumption 3 implies full dimensionality if the game is symmetric (Assumption 1). Therefore, for symmetric games, Assumption 3, the NEU condition, and the full dimensionality condition are all equivalent. Also observe that the only class of games Assumption 3 excludes is the games of pure coordination, in which all players' payoffs are the same whatever the action profile is (i.e., V^{**} has dimension 1).

The main theorem will be vacuous if $V^{**} = \emptyset$. So we will assume V^{**} is not empty, and then we have:

Proposition 1: *Suppose the game is symmetric (Assumption 1) and the set V^{**} is not empty. Then Assumption 3 implies the full dimensionality of V^{**}.*

Proof: Take a point v' in V^{**}. Then $v' \gg 0$, i.e. $v'_i > 0$ for all i. Now by Assumption 3, there exist $k, l \in I (k \neq l)$ and $a \in A$ such that $u_k(a) < u_l(a)$. Let $v = u(a)$. Then $v_k < v_l$. Think of the set W of all the permutations of v. That is, $W = \{\pi(v) : \pi \text{ is a permutation of } I\}$.

Since V is symmetric by Lemma 2, the set W is contained in V. Moreover, the convex hull of W, $co(W)$, is also contained in V since V is convex. Observe that $co(W)$ is at least $(n-1)$-dimensional since it contains all the permutations of v. In fact, it is $(n-1)$-dimensional because it is contained in the hyperplane $H = \{w \in I\!R^n : \sum_{i=1}^n w_i = \sum_{i=1}^n v_i\}$. Since the hyperplane H cannot contain both 0 and v', the convex hull of $co(W)$ together with 0 and v', which is $co(co(W), 0, v')$, is n-dimensional. This set is obviously contained in V since V is convex, giving us the conclusion that V is n-dimensional. *Q.E.D.*

3. THE FOLK THEOREM UNDER OBSERVABLE MIXED ACTIONS

In this section, we will prove a Folk Theorem under anonymity with the assumption that mixed actions are observable. That is, every feasible, strictly individually rational payoff vector is a sequential equilibrium outcome of the repeated game when players are sufficiently patient.

The intuition for the proof is as follows. The main observation in the proof is that, although players do not know who has deviated, they still know that a deviation has occurred and, in addition, whether he himself has deviated or not. This observation, together with the assumption of symmetry which implies that the punishing action can be constructed to be symmetric across the players, makes it possible for the punishers to coordinate to punish the deviator without knowing the identity of the deviator. The deviator, in turn, has an incentive to *cooperate* to the punishment since the punishment scheme is designed to punish the deviator (and any deviator in punishment phases) even without the knowledge of the identity of the deviator.

We say an action profile α_{-i} for players $j \neq i$ *symmetric* when the actions are the same for all players $j \neq i$. Then we have:

Lemma 3: *There exists a symmetric $\underline{\alpha}^i_{-i}$ such that $\underline{\alpha}^i = (\underline{\alpha}^i_i, \underline{\alpha}^i_{-i})$ is a minimax profile against player i.*

Proof: Let $\alpha = (\alpha_i, \alpha_{-i})$ be a minimax profile for i, i.e., $u_i(\alpha_i, \alpha_{-i}) = \underline{v}_i (= 0)$. Now let

$$\underline{\alpha}^i_{-i} = \frac{1}{m} \sum_{\alpha'_{-i} \in P(\alpha_{-i})} \alpha'_{-i} \, ,$$

where m is the number of elements in $P(\alpha_{-i})$. Then $\underline{\alpha}^i_{-i}$ is symmetric, and we have $u_i(\alpha_i, \underline{\alpha}^i_{-i}) = \frac{1}{m} \sum_{\alpha'_{-i} \in P(\alpha_{-i})} u_i(\alpha_i, \alpha'_{-i}) = \frac{1}{m} m u_i(\alpha_i, \alpha_{-i}) = u_i(\alpha_i, \alpha_{-i}) = \underline{v}_i (= 0)$ by Lemma 1. The action α_i is indeed a maximizer against $\underline{\alpha}^i_{-i}$: Suppose there is $\alpha'_i \neq \alpha_i$ such that $u_i(\alpha'_i, \underline{\alpha}^i_{-i}) > u_i(\alpha_i, \underline{\alpha}^i_{-i})$. But this implies $u_i(\alpha'_i, \alpha_{-i}) > u_i(\alpha_i, \alpha_{-i})$, which is a contradiction. *Q.E.D.*

We have the following two propositions which will be used in the proof of Theorem 1.

Proposition 2: *There exist $(\underline{\alpha}^i = (\underline{\alpha}^i_i, \underline{\alpha}^i_{-i}))_{i=1}^n$ such that*

(1) $\forall i, u_i(\underline{\alpha}^i) = \underline{v}_i (= 0)$.

(2) $\forall i, \underline{\alpha}^i_{-i}$ *is symmetric and*

(3) Each $\underline{\alpha}^i$ is a permutation of every other.

Proof: Let $\underline{\alpha}^1 = (\underline{\alpha}^1_1, \underline{\alpha}^1_{-1})$ be as in Lemma 3 for player 1. Now let $(\underline{\alpha}^i = (\underline{\alpha}^i_i, \underline{\alpha}^i_{-i}))_{i=1}^n$ be the permutations of it. *Q.E.D.*

Proposition 3: *For any $v \in V^{**}$, there exist $x^1, x^2, \ldots, x^n \in V^{**}$ such that for all i,*

(1) $x^i \ll v$.

(2) For some real $\epsilon > 0$, we have $x^i = (x, \ldots, x, x - \epsilon, x, \ldots, x)$ where $x - \epsilon$ is i-th component.

(3) There exists a symmetric $\tilde{\alpha}^i_{-i}$ such that $x^i = u(\tilde{\alpha}^i_i, \tilde{\alpha}^i_{-i})$.

Proof: Since $v \in V^{**}$ and V^{**} has full dimension (Proposition 1) and is symmetric (Lemma 2), there exist a vector $x = (x, \ldots, x) \in V^{**}$ and $\epsilon > 0$ such that every x' in the open ball $B(x, 2\epsilon)$ is strictly less than v ($x' \ll v$) and $x' \in V^{**}$. Now take $x^i = (x, \ldots, x, x - \epsilon, x, \ldots, x)$ where $x - \epsilon$ is i-th component. Then $x^i \in B(x, 2\epsilon)$. Thus what remains to prove is part (3).

We will prove for x^1. Since $x^1 \in V^{**}$, there exists $\gamma = (\gamma_1, \gamma_{-1})$ such that $x^1 = u(\gamma_1, \gamma_{-1})$. Now let

$$\tilde{\alpha}^1_{-1} = \frac{1}{m} \sum_{\gamma'_{-1} \in P(\gamma_{-1})} \gamma'_{-1}\,,$$

and $\tilde{\alpha}^1_1 = \gamma_1$. Then $u_1(\tilde{\alpha}^1_1, \tilde{\alpha}^1_{-1}) = \frac{1}{m} m u_1(\gamma_1, \gamma_{-1}) = u_1(\gamma_1, \gamma_{-1}) = x^1_1$ by Lemma 1. For $k \geq 2$, a moment's thought will convince the reader that $u_k(\tilde{\alpha}^1_1, \tilde{\alpha}^1_{-1}) = u_k(\gamma_1, \gamma_{-1}) = x^1_k$ since x^1 is symmetric in players $2, 3, \ldots, n$ and the game is symmetric (Assumption 1). x^k's for $k \geq 2$ are just permutations of x^1. *Q.E.D.*

Now we have our main theorem.

Theorem 1: (The Folk Theorem under Observable Mixed Actions.) *Suppose Assumptions 1,2, and 3 hold. Then, for any payoff vector v in V^{**}, there exists a discount factor $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is a sequential equilibrium payoff vector of the repeated game.*

Proof: Fix $v \in V^{**}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an action profile such that $u(\alpha) = v$. By Proposition 2, we can find minimax profiles $\underline{\alpha}^1, \ldots, \underline{\alpha}^n$ with the properties stated there. And by Proposition 3, we can also find payoff vectors $x^1, x^2, \ldots, x^n \in V^{**}$ with the properties stated there.

We will use the fact that even though players don't know exactly who has deviated when a deviation occurred, still each of them knows that a deviation has occurred by looking at the public outcome y^{t-1} and, in addition, whether he himself has deviated or not. For this purpose, let $y(\underline{\alpha}) = [\underline{\alpha}_1^i, \underline{\alpha}_2^i, \ldots, \underline{\alpha}_n^i]$. By construction, $y(\underline{\alpha})$ is the same for all i and

$$y(\underline{\alpha}) = [\underline{\alpha}_d, \underbrace{\underline{\alpha}_p, \ldots, \underline{\alpha}_p}_{(n-1) \text{ times}}].$$

Similarly,

$$y(\tilde{\alpha}) = [\tilde{\alpha}_d, \underbrace{\tilde{\alpha}_p, \ldots, \tilde{\alpha}_p}_{(n-1) \text{ times}}].$$

Now we will give a (repeated game) strategy for player i which will support v as a sequential equilibrium outcome.[7] This strategy will be described in a Markovian sense such that the game will proceed to one of the following three "states" depending on the current state and the public outcome in the previous period. For expositional clarity, we will imagine that every player keeps a private tag (known only to himself) until it is replaced by another. This tag is either "deviator", or "punisher", or "0". Initially, every player's tag is "0".

Start in state v.

state v**:** Play α_i.

[1] When y^{t-1} is different from $[\alpha_1, \ldots, \alpha_n]$ by exactly one component: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$. (If a player, say j, has deviated, then y^{t-1} is different from

[7] The belief part of the sequential equilibrium is implied by the description of the strategy in an obvious way.

$[\alpha_1, \ldots, \alpha_n]$ by exactly one component, and so everybody knows that the tags are updated and also what his own updated tag is.)

[2] In all other cases: Stay in state v.

state $y(\underline{\alpha})$**:** If his tag is "deviator", then play $\underline{\alpha}_d$. If his tag is "punisher", then play $\underline{\alpha}_p$.

[1] When y^{t-1} is different from $y(\underline{\alpha})$ by exactly one component: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$. (Player i might have had a tag of "deviator" or "punisher" when the deviation occurred. As before, players know the fact that the tags are updated and what their own updated tags are.)

[2] In all other cases: With probability q, stay in state $y(\underline{\alpha})$ and with probability $1-q$, go to state $y(\tilde{\alpha})$.

state $y(\tilde{\alpha})$**:** If his tag is "deviator", then play $\tilde{\alpha}_d$. If his tag is "punisher", then play $\tilde{\alpha}_p$.

[1] When y^{t-1} is different from $y(\tilde{\alpha})$ by exactly one component: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[2] In all other cases: Stay in state $y(\tilde{\alpha})$.

Now we show that no one-shot deviation from the specified strategy by any player in any state is profitable.[8] Define $b = \max_i \max_{a \in A} u_i(a)$ and $w = u_2(\underline{\alpha}^1)$. And also take $0 \leq q < 1$ such that

$$b < \frac{2-q}{1-q}(x - \epsilon) \tag{$*$}$$

holds.

state $y(\underline{\alpha})$: The deviator obviously does not have an incentive to deviate again since, by deviating, he will get at most 0 in that period and go back to state $y(\underline{\alpha})$ with the same

[8] The verification of this strategy below closely follows Abreu, Dutta, and Smith (1994).

tag ("deviator") next period. More specifically, if he conforms to the specified strategy, he will get $\underline{W}_d = \delta(q\underline{W}_d + (1-q)(x-\epsilon))$, i.e.,

$$\underline{W}_d = \frac{\delta(1-q)}{1-\delta q}(x-\epsilon).$$

($\underline{W}_d$ is the repeated game average payoff when in state $y(\underline{\alpha})$.) But if he deviates, he will at most get $\delta\underline{W}_d$. Thus, he has no incentive to deviate.

For a punisher, if he conforms, his average payoff is

$$\frac{1-\delta}{1-\delta q}w + \frac{\delta(1-q)}{1-\delta q}x.$$

If he deviates, he gets at most

$$b + \delta\underline{W}_d = (1-\delta)b + \delta\frac{\delta(1-q)}{1-\delta q}(x-\epsilon).$$

As $\delta \to 1$, the former expression goes to x, while the latter to $x-\epsilon$. Thus, for sufficiently large δ, punishers do not have an incentive to deviate.

state $y(\tilde{\alpha})$: First, for the deviator. The maximum gain (in average payoff) from deviating in this state is

$$\begin{aligned}
&(1-\delta)b + \delta\underline{W}_d - (x-\epsilon) \\
=&(1-\delta)[b - \frac{1+\delta-\delta q}{1-\delta q}(x-\epsilon)].
\end{aligned}$$

For δ near 1, this expression is negative due to (*). So, the deviator does not want to deviate. It is also clear that the punishers don't want to deviate either, since the payoff a punisher receives (x) is greater than that of a deviator ($x-\epsilon$) by construction.

state v: For each player i,

$$\begin{aligned}
v_i >&(1-\delta)b + \delta\underline{W}_d \\
=&(1-\delta)b + \delta\frac{\delta(1-q)}{1-\delta q}(x-\epsilon)
\end{aligned}$$

for large δ, since $v_i > x-\epsilon$ by construction.

We have shown that no player has an incentive to make a one-shot deviation from any state. Then, the reader can easily see that no (possibly infinite) sequence of deviations is profitable. Therefore, the proposed strategy profile is a sequential equilibrium.[9] *Q.E.D.*

4. THE FOLK THEOREM UNDER UNOBSERVABLE MIXED ACTIONS

In this section, we need to make a slight change in our notation: the public outcome y^t is no longer a list $[\alpha_1^t, \ldots, \alpha_n^t]$ of mixed action profile, but a realization of it, say $[a_1^t, \ldots, a_n^t]$ in which a_i^t's are pure actions. The only significant use of the observable mixed actions assumption is when we suppose the minimaxing actions are observable. For other cases, we can replace any particular mixed action profile with a deterministic sequence of pure action profiles. Consequently, what we prove below is mainly about how to make the punishers in the minimaxing stage be indifferent among the pure actions in the support of the minimaxing action $\underline{\alpha}_p$. First we have the following corollary of Proposition 3.

Corollary 1: *For any $v \in V^{**}$, there exist $x^1, x^2, \ldots, x^n \in V^{**}$ and $x_\epsilon^1, x_\epsilon^2, \ldots, x_\epsilon^n \in V^{**}$ such that for all i,*

(1) $x^i \ll v$, $x_\epsilon^i \ll v$.

(2) For some real $\epsilon > 0$, we have $x^i = (x, \ldots, x, x-\epsilon, x, \ldots, x)$ and $x_\epsilon^i = (x+\epsilon, \ldots, x+\epsilon, x-\epsilon, x+\epsilon, \ldots, x+\epsilon)$ where $x-\epsilon$ is i-th component.

(3) There exists a symmetric $\tilde{\alpha}_{-i}^i$ such that $x^i = u(\tilde{\alpha}_i^i, \tilde{\alpha}_{-i}^i)$.

(4) There exists a symmetric $\hat{\alpha}_{-i}^i$ such that $x_\epsilon^i = u(\hat{\alpha}_i^i, \hat{\alpha}_{-i}^i)$.

Proof: The proof of Proposition 3 implies this. *Q.E.D.*

---

[9] A keen reader would observe that there exists a history in which more than one player simultaneously deviated, but still the public outcome resulting from it is identical to the one resulting from a unilateral deviation. Then the strategy proposed above may not be optimal; some players may have better knowledge about the actual history than others, and hence they may exploit it. We simply point out that there is *a* sequential equilibrium following it. Observe that no sequence of unilateral one-shot deviations will lead to such a history.

Theorem 2: (The Folk Theorem under Unobservable Mixed Actions) *Suppose Assumptions 1,2, and 3 hold. Then, for any payoff vector v in V^{**}, there exists a discount factor $\underline{\delta} < 1$ such that, for all $\delta \in (\underline{\delta}, 1)$, v is a sequential equilibrium payoff vector of the repeated game.*

Proof: Fix $v \in V^{**}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an action profile such that $u(\alpha) = v$. By Proposition 2, we can find minimax profiles $\underline{\alpha}^1, \ldots, \underline{\alpha}^n$ with the properties stated there. And by Corollary 1, we can also find payoff vectors $x^1, x^2, \ldots, x^n \in V^{**}$ and $x^1_\epsilon, x^2_\epsilon, \ldots, x^n_\epsilon \in V^{**}$ with the properties stated there. We can, without loss of generality, assume α, $(\tilde{\alpha}^i)_{i \in I}$, $(\hat{\alpha}^i)_{i \in I}$ are all pure action profiles. Moreover, we can choose $\underline{\alpha}^i_i$, i.e., $\underline{\alpha}_d$ to be a pure action.

Now suppose $\underline{\alpha}^i_j$ for $j \neq i$, i.e., $\underline{\alpha}_p$ is a nontrivial mixed action. Let $a_1, \ldots, a_m$ be the pure actions used in $\underline{\alpha}_p$, with the property that

$$u_j(a_{k-1}; \underline{\alpha}_d, \underbrace{\underline{\alpha}_p, \ldots, \underline{\alpha}_p}_{(n-2) \text{ times}}) \leq u_j(a_k; \underline{\alpha}_d, \underbrace{\underline{\alpha}_p, \ldots, \underline{\alpha}_p}_{(n-2) \text{ times}})$$

for all $k = 2, \ldots, m$ (a_{k-1} and a_k are player j's actions).

Let $r_k = u_j(a_k; \underline{\alpha}_d, \underline{\alpha}_p, \ldots, \underline{\alpha}_p) - u_j(a_1; \underline{\alpha}_d, \underline{\alpha}_p, \ldots, \underline{\alpha}_p)$. Then r_k can be interpreted as a gain from playing a_k instead of a_1. Let N_k be the total number of a_k in y^{t-1}, the realization of $y(\underline{\alpha})$ in period $t-1$. Then each y^{t-1} induces a vector $(N_1, \ldots, N_m)$ with $N_1 + \ldots + N_m = n - 1$. (If $\underline{\alpha}_d$, which is a pure action, is a_k in the support of $\underline{\alpha}_p$, then subtract 1 from N_k.) Let $z_0 = (n-1, \underbrace{0, \ldots, 0}_{(m-1) \text{ times}})$ and for any vector $z = (N_1, \ldots, N_m)$, calculate $S_z = r_2 N_2 + r_3 N_3 + \ldots + r_m N_m$. Now for each z, define a probability p_z by

$$(1 - \delta) S_z = \delta (p_{z_0} - p_z) \epsilon. \qquad (**)$$

As $\delta \to 1$, there exists $\{p_z\}$ which satisfy $(**)$.

As in Theorem 1, let

$$y(\underline{\alpha}) = [\underline{\alpha}_d, \underbrace{\underline{\alpha}_p, \ldots, \underline{\alpha}_p}_{(n-1) \text{ times}}], \quad y(\tilde{\alpha}) = [\tilde{\alpha}_d, \underbrace{\tilde{\alpha}_p, \ldots, \tilde{\alpha}_p}_{(n-1) \text{ times}}].$$

Also let

$$y(\hat{\alpha}) = [\hat{\alpha}_d, \underbrace{\hat{\alpha}_p, \ldots, \hat{\alpha}_p}_{(n-1) \text{ times}}].$$

The following is a sequential equilibrium strategy for i which supports v. As before, every player's tag is "0" initially.

Start in state v.

state v**:** Play α_i.

[1] When y^{t-1} is different from $[\alpha_1, \ldots, \alpha_n]$ by exactly one component: If player i himself is the deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[2] In all other cases: Stay in state v.

state $y(\underline{\alpha})$**:** If his tag is "deviator", then play $\underline{\alpha}_d$. If his tag is "punisher", then play $\underline{\alpha}_p$.

case 1. $\underline{\alpha}_d = a_k$ for some $k = 1, \ldots, m$.

[1] When no component of y^{t-1} lies outside of the support of $\underline{\alpha}_p$, with the induced vector z: With probability q stay in state $y(\underline{\alpha})$, with probability $1 - q - p_z$ go to state $y(\tilde{\alpha})$, and with probability p_z go to state $y(\hat{\alpha})$.

[2] When one component of y^{t-1} lies outside of the support of $\underline{\alpha}_p$: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[3] In all other cases: With probability q stay in state $y(\underline{\alpha})$ and with probability $1 - q$ go to state $y(\tilde{\alpha})$.

case 2. $\underline{\alpha}_d \neq a_k$ for all $k = 1, \ldots, m$.

[1] When one component of y^{t-1} lies outside of the support of $\underline{\alpha}_p$ and that component is $\underline{\alpha}_d$, with the induced vector z: With probability q stay in state $y(\underline{\alpha})$, with probability $1 - q - p_z$ go to state $y(\tilde{\alpha})$, and with probability p_z go to state $y(\hat{\alpha})$.

[2] When one component of y^{t-1} lies outside of the support of $\underline{\alpha}_p$ and that component is different from $\underline{\alpha}_d$, or no component lies outside of the support of $\underline{\alpha}_p$: Go to state $y(\underline{\alpha})$ with the ongoing tags.

[3] When two components of y^{t-1} lie outside of the support of $\underline{\alpha}_p$ and that one of them is $\underline{\alpha}_d$: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[4] In all other cases: With probability q stay in state $y(\underline{\alpha})$ and with probability $1-q$ go to state $y(\tilde{\alpha})$.

state $y(\tilde{\alpha})$**:** If his tag is "deviator", then play $\tilde{\alpha}_d$. If his tag is "punisher", then play $\tilde{\alpha}_p$.

[1] When y^{t-1} is different from $y(\tilde{\alpha})$ by exactly one component: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[2] In all other cases: Stay in state $y(\tilde{\alpha})$

state $y(\hat{\alpha})$**:** If his tag is "deviator", then play $\tilde{\alpha}_d$. If his tag is "punisher", then play $\tilde{\alpha}_p$.

[1] When y^{t-1} is different from $y(\hat{\alpha})$ by exactly one component: If player i himself is a deviator, then update his tag to "deviator"; otherwise update his tag to "punisher". Go to state $y(\underline{\alpha})$.

[2] In all other cases: Stay in state $y(\hat{\alpha})$.

Since we assumed that α_i, $\tilde{\alpha}_d$, $\tilde{\alpha}_p$, $\hat{\alpha}_d$, and $\hat{\alpha}_p$ are all pure actions, the proof in Theorem 1 passes over in those states. All we need to show is that punishers are indifferent among $a_1, \dots, a_m$ so that they will play the mixed action α_p.

The punisher j's average payoff from a_k $(k = 1, \dots, m)$ is

$$(1-\delta)u_j(a_k; \underline{\alpha}_d, \underline{\alpha}_p, \dots, \underline{\alpha}_p) + \delta\Big\{q\big[\frac{1-\delta}{1-\delta q}w + \frac{\delta(1-q)}{1-\delta q}x\big] + (1-q-p_z)x + p_z(x+\epsilon)\Big\}$$

Playing k' instead of k will result in the change of this payoff by the amount of

$$(1-\delta)(r_{k'} - r_k) + \delta(p_{z'} - p_z)\epsilon,$$

where z' is a vector $(N'_1, \ldots, N'_m)$ which is different from $z = (N_1, \ldots, N_m)$ only in that $N'_{k'} = N_{k'} + 1$ and $N'_k = N_k - 1$. So, $S_{z'} - S_z = r_{k'} - r_k$. Then we have

$$
\begin{aligned}
&(1-\delta)(r_{k'} - r_k) + \delta(p_{z'} - p_z)\epsilon \\
=&(1-\delta)(r_{k'} - r_k) + \delta(p_{z_0} - p_z)\epsilon - \delta(p_{z_0} - p_{z'})\epsilon \\
=&(1-\delta)(r_{k'} - r_k) + (1-\delta)S_z - (1-\delta)S_{z'} \qquad \text{(by (**))} \\
=&(1-\delta)(r_{k'} - r_k) - (1-\delta)(r_{k'} - r_k) \\
=&0.
\end{aligned}
$$

Thus punishers are indifferent among $a_1, \ldots, a_m$. *Q.E.D.*

5. DISCUSSION AND RELATED LITERATURE

The framework of the present paper is subsumed in the very general framework of Fudenberg, Levine, and Maskin (1994, FLM hereafter). Specifically, the unordered action profile $[\alpha_1, \ldots, \alpha_n]$ is a public signal in FLM. Then it is easy to verify that the present structure satisfies the pairwise full rank (Condition 6.2.) and the individual full rank (Condition 6.3.) of FLM. Therefore, the Folk Theorem of FLM applies.

The present paper, however, extends FLM's result in three nontrivial ways.[10] First, their framework is confined to the finite pure actions case. In contrast, our result can be applied equally well to a continuum of actions case, which is often the case in many economic situations: Interpret the set $\Delta(A_i)$ of mixed actions as the set of pure actions. Then Theorem 1 of the present paper covers this case exactly. Second, their result pertains only to the interior of the set V^{**} of feasible, strictly individually rational payoff vectors. In particular, efficient payoff vectors are not shown to be supported. Since the interest in Folk Theorems arises largely due to the possibility of efficient outcomes, it would be nice to have a result supporting them. We prove here a Folk Theorem which applies to

[10] This is not to say that we extend FLM's result for the general environments. We extend their result for this special case of anonymity.

the whole set V^{**}. Lastly, and perhaps most importantly, we provide explicit repeated game strategies which are simple and intuitive, while they are not directly concerned with strategies. As we explained in the Introduction, we believe that what matters more in Folk Theorems is not *whether* people can sustain cooperation but *how* they can.

Green (1980), Sabourian (1990), and Levine and Pesendorfer (1995) show that if the anonymity assumption is incorporated by assuming that players can only observe some *noisy* aggregate public outcome (like prices), then the set of repeated game equilibria shrinks to the set of stage game Nash equilibria as the number of players increases to infinity provided some condition is imposed on the aggregate public outcome. This condition is that, roughly speaking, the informativeness of the outcome tends to be negligible as each player becomes arbitrarily small. The present paper differs from the papers mentioned in that we assume a deterministic, rather than noisy, public outcome about the past play, which is $y^t = [\alpha_1^t, \ldots, \alpha_n^t]$, and that we are concerned with determining the *whole set* of payoff vectors which can be supported as sequential equilibrium outcomes of the repeated game. Observe that first, as mentioned in section 2, we can easily extend our results to cases where the public outcome is constrained to be aggregate, say $y^t = \sum_i \alpha_i^t$ and second, our result is independent of the number of players.

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